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FUNDAMENTAL SOLUTIONS AND NUMERICAL METHODS FOR FLOW PROBLEMS*

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SUMMARY

An approach for the numerical solution of flow problems based on the concept of fundamental solutions of differential equations is described. This approach uses the finite element methodology but does not rely on the concept of variational principle or that of residuals. The approach is shown to be well-suited for many types of flow problems. Various applications of this approach are discussed in this paper, with particular emphasis placed on the solution of potential flows and viscous flows containing appreciable regions of separation.

KEY WORDS Potential Flows Navier–Stokes Problems

INTRODUCTION

During the past decade, the fluid dynamics community has experienced an intense growth of interest in the computational simulation of flow phenomena. This growth of interest is a result of the impressive progress made in the development of efficient and accurate numerical procedures for the solution of certain types of flows. At present, these type of flows are routinely computed in engineering applications. Intensive research efforts are under way to develop routine capabilities for computing other types of flows whose computations require excessive computer time.

Until relatively recently, the finite-difference approach has been emphasized by researchers developing computational methods for viscous flows. For boundary layer flows, the finite-difference approach has been found to be particularly powerful. There exist today a number of accurate and efficient finite-difference codes with which boundary layer flows of various types, including three-dimensional and turbulent flows, can be solved routinely and economically on computers with wide accessibility. In contrast, the development of finite-difference procedures for general viscous flows, in which flow separation is an important feature, has experienced only limited success. In the numerical solution of such flows, the full Navier–Stokes equations must be computed in at least the separated region of the flow. In this paper, such flows are referred to as Navier–Stokes flows.

Finite-difference solutions of a small number of Navier–Stokes flows appeared in the 1950s and 1960s. These earlier solutions, however, were limited to very low Reynolds numbers and to extremely simple flow geometries. By the beginning of 1970, it became evident that, in order to free the numerical procedure from the two limitations just stated, not only must faster and more powerful computers be built, but also innovative numerical

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methods must be developed. Since most applications of practical importance involve complex flow geometries and high Reynolds numbers, the removal of these two limitations became a focal point of computational fluid dynamics research in the 1970s.

Extensive developments of grid generation techniques were undertaken by many researchers in the 1970s. These developments were directed mainly towards the use of finite-difference procedures. By the end of the 1970s, the development of grid-generation techniques that produce body-fitted grids for two-dimensional problems had approached a state of maturity. Currently, considerable efforts are in progress to develop grid generation techniques for three-dimensional applications.

The use of body-fitted grids for two-dimensional Navier–Stokes flows did not free the finite-difference procedures from their Reynolds number limit. It was found in practice that, with increasing Reynolds number, the amount of computation required to solve a problem increases rapidly. At the same time, the accuracy of the solution deteriorates rapidly. Various techniques were devised to improve the computational efficiency of finite-difference procedures. In relation to grid systems, 'expanding' grids, i.e. grids with increasing spacing between grid lines as the distance from solid boundary increases, are commonly used. Since gradients of flow variables are usually large near the solid boundary and small far from it, an expanding grid requires a smaller number of grid points than that required by a non-expanding grid. The amount of computation required for high Reynolds number Navier–Stokes flows is therefore reduced with the use of an expanding grid. This reduction, however, is not sufficient to remove the Reynolds number limit of the finite-difference procedures.

Relatively recently, finite-element procedures have received some emphases in the fluid dynamics community. These emphases are motivated by the persistent need in fluid dynamics for alternatives to the finite-difference approach. An impetus for an accelerated interest in the finite-element approach was provided by the success of this approach in the field of solid mechanics. Many researchers were of the opinion that the application of the finite-element approach to flow problems represents a natural extension of the approach to solid mechanics problems. Such an extension, however, has proved to be not straightforward and only some of the difficulties experienced in using the finite difference approach has been removed by the finite element approach. In particular, the finite-element approach has not substantially reduced the excessive computing needs of the finite-difference techniques, nor has it removed the difficulty of obtaining accurate solutions for high Reynolds number Navier–Stokes flows.

In retrospect, the familiar finite-element procedures should not have been expected to remove all the difficulties experienced by the finite-difference procedures. It is known that, if finite-element nodes are arranged in a uniformly spaced rectangular array, then the finite-element algebraic equations obtained through the concept of the variational principle or that of residuals, e.g. the Galerkin's concept, are often identical to finite-difference equations obtained using the same grid systems. The familiar finite-element approach is therefore not as fundamentally different from the finite-difference approach as some researchers have claimed it to be.

More than a decade ago, this author initiated a research programme with the goal of removing some of the difficulties experienced in computing high Reynolds number Navier–Stokes flows. During the initial stage of this research programme, an approach which uses the finite-element methodology but which does not rely on the concept of the variational principle or that of residuals was found. This approach possesses several highly desirable attributes. With this approach, the differential equations of motion are recast into the form of integral representations through the use of the concept of fundamental solutions. The integral representations are then solved numerically.

The integral representations are completely equivalent to the more familiar differential equations describing the flows. They contain no assumptions or approximations other than those contained in the original differential equations. Each of the integral representations is composed of an integral over the fluid domain and an integral over the boundary of the fluid domain. The integral representation approach is therefore a generalized version of the boundary-element method that deals with the boundary integrals.

During the past decade, this author and his co-workers have carried the development of the integral-representation approach through several stages. In each stage of development, several distinguishing features of the approach were incorporated into the solution procedure and substantial improvements in both solution efficiency and solution accuracy were achieved. The development of this approach has now reached a reasonable stage of completion. In particular, Navier–Stokes flows have been treated extensively and the advantages offered by the integral-representation method are well documented for these type of flows. It has been demonstrated, in addition, that the method is also well-suited for the other types of flow problems.

In this paper, the fundamental concepts of the integral-representation approach are described. Applications of this approach to Navier–Stokes flows are discussed. The boundary-element methods for potential flows are shown to be a specialized application of the integral-representation approach. Numerical illustrations of the application of this approach to Navier–Stokes flows are presented. In addition, the interplay between the physical and the numerical aspects of flow problems, brought into focus by the use of the integral representation approach, is discussed.

FUNDAMENTAL SOLUTIONS

The mathematical foundation of the integral-representation approach is the fundamental solution, also called the principal solution, of differential equations. To demonstrate the solution procedure and several unique features of the approach, the relatively simple problem of numerical solution of Poisson's equation is considered below. Two additional types of differential equations relevant to flow problems are discussed together with their associated fundamental solutions and the corresponding integral representations in the Appendix of this paper.

Corresponding to a specific differential equation, there are generally several different forms of integral representation possible. The equivalence of one form to another is not always obvious. The integral representations presented in the Appendix are in forms well suited for tthe computation of flow problems.

Consider Poisson's equation in the form

$$\nabla^2\phi = g, \text{ in the Region } R \tag{1}$$

where ϕ is the function to be determined in the region R and the inhomogeneous term g can be a function of position and field variables. The region R is either singly- or multiply-connected and is bounded by B, which consists of either a single closed boundary or a closed outer boundary and one or more closed inner boundaries. An integral representation for the function ϕ is[1]

$$\phi(\mathbf{r}) = \int_R g_0 P \, dR_0 + \oint_B \left(\phi_0 \frac{\partial P}{\partial n_0} - P\frac{\partial \phi_0}{\partial n_0}\right) dB_0 \tag{2}$$

where $\mathbf{r}$ is the position vector, n is the outward normal direction, the subscript '0' indicates a variable, or a differentiation, or an integration in the $\mathbf{r}_0$ space. The function P is the principal

solution defined by

$$P(\mathbf{r}, \mathbf{r}_0) = \begin{cases} -\dfrac{1}{4\pi r'}, & \text{in three-dimensional problems} \\ -\dfrac{1}{2\pi} \ln \dfrac{r}{r'}, & \text{in two-dimensional problems} \end{cases} \tag{3}$$

where r' is the magnitude of $\mathbf{r}'$ defined by

$$\mathbf{r}' = \mathbf{r}_0 - \mathbf{r} \tag{4}$$

The principal solution P satisfies Laplace's equation $\nabla^2 P = 0$. The singularity of P at $\mathbf{r} = \mathbf{r}_0$ such that

$$\int_R \nabla^2 P \, dR = \begin{cases} 1 & \text{if } R \text{ contains the point } \mathbf{r} = \mathbf{r}_0 \\ 0 & \text{if } R \text{ does not contain the point } \mathbf{r} = \mathbf{r}_0 \end{cases} \tag{5}$$

Consequently, the Laplacian of equation (2) gives equation (1).

Suppose that both ϕ and $\partial\phi/\partial n$ are known on the boundary B, then a quadrature of the right hand side of equation (2) determines the function ϕ everywhere in R. As is well known, however, the correct boundary condition for equation (1) is either Dirichlet's or Neumann's, or a linear combination of the two, over the entire boundary B. The use of equation (2) in place of equation (1) appears to require a knowledge of both ϕ and $\partial\phi/\partial n$ on B and therefore presents a difficulty. The accepted procedure to remove this difficulty is to modify the principal solution in such a way that the modified solution, called Green's function, satisfies Laplace's equation and possesses a singularity at $\mathbf{r} = \mathbf{r}_0$, characterized by equation (5). Green's function, in addition, is required to have the property that either its value or its normal gradient vanishes on the boundary B. The integral representation for ϕ expressed in terms of the Green's function is identical in form to equation (2), but the integrand in the boundary integral now contains only one term, involving either $\partial\phi/\partial n$ or ϕ. Obviously, each Green's function represents a solution of a differential equation associated with a specific boundary geometry. As a consequence, methods of solution using Green's function require that a Green's function be found for each boundary geometry of interest. For a few relatively simple geometries, Green's functions are obtainable analytically by using the method of images or other special techniques. Most flow problems of practical importance, however, involve boundary geometries of such complexity that Green's functions are not obtainable analytically.

The fundamental solution P, as defined by equation (3), may be considered a special case of Green's function which vanishes at infinity. The fact that the fundamental solution is independent of any specific boundary geometry makes it well-suited for numerical procedures. Regarding the proper prescription of boundary conditions, a closer examination of equation (2) reveals that it is possible to use this equation in establishing the missing information on the boundary B, using properly prescribed information. Consider the case where the value of ϕ is prescribed on B. In order to obtain the normal derivative $\partial\phi/\partial n$ on B, equation (2) is applied at the boundary, yielding, upon rearrangement, an integral equation with $\partial\phi/\partial n$ as the unknown function

$$\oint_B P_B \frac{\partial \phi_0}{\partial n_0} \, dB_0 = F(\mathbf{r}_B) \tag{6}$$

where $\mathbf{r}_B$ is a point on the boundary B, $P_B = P(\mathbf{r}_0, \mathbf{r}_B)$, and

$$F(\mathbf{r}_B) = \int_R g_0 P_B \, dR_0 + \oint_B \phi_0 \frac{\partial P_B}{\partial n_0} dB_0 - \phi(\mathbf{r}_B) \tag{7}$$

The integrands of equation (7) are known and $\phi(\mathbf{r}_B)$ is prescribed. Thus $F(\mathbf{r}_B)$ is determinate. The theory of solution of the integral equation (6) is discussed extensively and fundamental existence and uniqueness theorems are available in several well known treatises.[2,3] For fluid flow applications, these questions are most conveniently and fruitfully studied in conjunction with each particular type of flow problem at hand.

Quite often in flow problems, the prescribed condition on B is $\partial\phi/\partial n$. In order to establish the missing information, ϕ, on the boundary B, equation (2) is applied at the boundary. The resulting integral equation contain ϕ_B as the unknown function. The solution of this integral equation gives ϕ_B and enables the computation of ϕ in R, away from B, using equation (2).

DISTINCTIVE CHARACTERISTICS

In its current context, a finite-element method is a numerical method for solving field equations through the mapping of a solution region into subregions, i.e. elements, each associated with a finite number of nodes. Continuum field variables are approximated in each element by element approximation functions. The field equations are then approximated by a set of algebraic equations containing unknown nodal values of the field variables. These unknown values are computed by solving the algebraic equations. When the problem of interest is expressed as a differential equation or a system of differential equations, current finite-element methods require that the problem be first recast into the form of integral relations. The integrals over the solution regions are replaced by sums of element integrals, each over an individual element. The element integrals, with suitably chosen element approximation functions, are evaluated analytically and expressed algebraically in terms of the co-ordinates of the nodes and the values of the field vaariables at the nodes. The finite-element approach offers an inherent flexibility in the selection of node locations.

Consider again the relatively simple problem of numerical solution of a scalar Poisson's equation (1). Equation (1) is equivalent to equation (2), which is expressed by the use of the finite-element techniques in the form

$$\phi_i = \sum_{j=1}^{Q} A_{ij} g_j + \sum_{k=1}^{S} \left[B_{ik} \phi_k - C_{ik} \left(\frac{\partial \phi}{\partial n} \right)_k \right] \tag{8}$$

where i refers to a node at which the value of ϕ is to be computed; j refers to a node in the region R; k refers to a node located on the boundary B; Q is the total number of nodes in the solution field, S is the number of nodes on the boundary B; A_{ij}, B_{ik} and C_{ik} are geometrical functions dependent only on the space co-ordinates of the node i and of the elements containing the nodes j or k, as the case may be. It is noted that, because of the singularity of the fundamental solution, the coefficients A_{ii}, B_{ii} and C_{ii} require special treatment. Such treatment is generally not difficult, although a limiting procedure is usually involved. It should be noted that with equation (2), or other integral representations presented in the Appendix, the integral over the boundary is not evaluated as a principal value. Rather, the point $\mathbf{r}$ is always in the region R of interest and this point only approaches B in the limit.

The solution procedure using equation (8) is as follows. If the boundary condition is Dirichlet's, then equation (8) is applied at the S boundary nodes, where ϕ_i is known, to yield

a system of S algebraic equations containing the S unknown values of $(\partial\phi/\partial n)_k$, $k = 1, 2, 3, \ldots S$. This system of equations may contain a defect and therefore requires an auxiliary condition for the solution of $(\partial\phi/\partial n)_k$. A method of establishing the auxiliary condition is discussed by Wu.[12]

Once the gradients $(\partial\phi/\partial n)_k$ are determined, they are placed in equation (8), which can now be used to evaluate explicitly, node by node, the values of ϕ at nodes away from the boundary B. The procedure is explicit if g is a known function of space co-ordinates or, more generally, a function of space co-ordinates and field variables that are known at each node j.

The ability to evaluate field variables explicitly distinguishes the integral-representation approach from other numerical procedures that are not based on the concept of fundamental solutions. It is known that finite-element procedures using the variational principle or the concept of residuals to establish integral relations, like finite-difference procedures, lead to implicit algebraic equations that approximate elliptic differential equations. The explicitness of the integral-representation approach has far reaching consequences in the computation of potential and Navier–Stokes flows. Some of these consequences are described in this paper. With regard to potential flow computations, the integral representation approach is central to the boundary-element methods that are receiving considerable attention in the current literature.

FLOW EQUATIONS

The time dependent motion of an incompressible viscous fluid is governed by the law of mass conservation and Newton's laws of motion. The mathematical statements of these laws are familiarly expressed in terms of the velocity vector $\mathbf{v}$ and the pressure p. They are known as the continuity and the Navier–Stokes equations:

$$\nabla \cdot \mathbf{v} = 0 \tag{9}$$

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} \tag{10}$$

where $\mathbf{v}$, p, ρ and ν are respectively the velocity, the pressure, the density, and the kinematic viscosity of the fluid; t is the time. For simplicity, ν is considered uniform in the present paper.

In principle, equations (9) and (10) are sufficient for the determination of p and $\mathbf{v}$ everywhere in the fluid domain at every time level, provided that the velocity values are prescribed initially in the domain and are also prescribed on the boundary of the domain at every subsequent time level. It is more convenient, however, to partition the problem into a kinematic part and a kinetic part through the introduction of the vorticity vector $\boldsymbol{\omega}$ defined by

$$\nabla \times \mathbf{v} = \boldsymbol{\omega} \tag{11}$$

Kinematics

Equations (9) and (11) together describe the kinematic aspect of the viscous flow problem. That is, they express the relationship between the vorticity field at any given instant of time and the velocity field at the same instant.

Two features of the kinematic differential equations are of particular significance. First, the differential equations are linear. The kinematic aspect of the problem is therefore amenable to classical mathematical analyses. Secondly, the stress–strain relation, which distinguishes a

solid from a fluid, does not enter into the kinematic aspect. The fluid and the solid in contact with it may therefore be treated together as one kinematic system and the mathematical analysis becomes particularly simple.

Kinetics

The kinetic aspect of the problem is described by equation (2). This equation is non-linear. With the concept of vorticity introduced, equation (2) is expressible as

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla h + \mathbf{v} \times \boldsymbol{\omega} - \nu \nabla \times \boldsymbol{\omega} \tag{12}$$

where h is the total head defined by

$$h = \frac{p}{\rho} + \tfrac{1}{2} v^2 \tag{13}$$

For steady flows, the term $\partial \mathbf{v}/\partial t$ vanishes, and equation (12) becomes

$$\nabla \times \boldsymbol{\omega} = \frac{1}{\nu} (\mathbf{v} \times \boldsymbol{\omega} - \nabla h) \tag{14}$$

Since $\boldsymbol{\omega}$ is the curl of $\mathbf{v}$, the divergence of $\boldsymbol{\omega}$ is identically zero. That is

$$\nabla \cdot \boldsymbol{\omega} = 0 \tag{15}$$

The set of equations (14) and (15) is analogous to the set of equations (11) and (9).

For time-dependent flows, by taking the curl of each term in equation (13), one obtains the following equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -(\mathbf{v} \cdot \nabla)\boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} + \nu \nabla^2 \boldsymbol{\omega} \tag{16}$$

Equation (16) is known as the vorticity transport equation. The term on the right-hand side of equation (16) represents the physical processes of convection, stretching and rotation, and diffusion of vorticity in a flow.

POTENTIAL FLOW AND BOUNDARY ELEMENT METHOD

Under general circumstances, the kinematic and the kinetic aspects of the flow problems are interlaced, and both aspects must be treated in the computation procedure. In the special case of potential flows, however, the vorticity field is everywhere zero away from the immediate vicinity of the solid boundary, and the kinetics of the flow problem needs not enter the solution procedure.

A potential flow is often thought of as an inviscid flow. In reality, it is more appropriate to consider a potential flow as an approximation of a high Reynolds number flow involving no appreciable region of flow separation. In such a flow, the vorticity, which must originate from the solid surface, can only diffuse a short distance into the interior of the fluid domain before being carried away by the fluid through the process of convection. The region of non-zero vorticity, and hence also the effects of viscosity is then confined to thin boundary layers adjacent to solid surfaces and thin wakes trailing the solid. For certain types of flows, the effects of the vortical wake on the flow near the solids are negligible. The vorticity in the thin boundary layers can be represented kinematically by a vortex sheet. The strength of this

vortex sheet is the integrated vorticity value across the boundary layer and is dependent on the location on the solid surface. This strength can be determined by using a boundary element method.

Consider equations (9) and (11). From equation (39) of the Appendix, one obtains an integral representation for the velocity vector

$$\mathbf{v}(\mathbf{r}) = -\int_R \boldsymbol{\omega}_0 \times \nabla_0 P \, dR_0 + \oint_B [\mathbf{v}_0 \cdot \mathbf{n}_0 - (\mathbf{v}_0 \times \mathbf{n}_0) \times] \nabla_0 P \, dB_0 \tag{17}$$

Consider a steady flow past the exterior of a finite solid body. For this flow, the boundary B is composed of the solid surface S and an external surface S_∞ enclosing S and infinitely far from S. In a reference frame attached to the solid, the velocity values on S is zero because of the no-slip condition. The contribution of the boundary integral in equation (17) to the velocity field in R is therefore entirely due to the surface S_∞. It has been shown[9] that this contribution is simply $\mathbf{v}_\infty$, the velocity of the free stream. One thus has

$$\mathbf{v}(\mathbf{r}) = -\int_R \boldsymbol{\omega}_0 \times \nabla_0 P \, dR_0 + \mathbf{v}_\infty \tag{18}$$

For the potential flow problem under consideration, with the vorticity in the boundary layers represented by a vortex sheet, one has

$$\mathbf{v}(\mathbf{r}) = -\oint_{S^+} \boldsymbol{\omega}_0 \times \nabla_0 P \, dR_0 + \mathbf{v}_\infty \tag{19}$$

The integrand of the integral in equation (19) can be rewritten as $\nabla \times (\boldsymbol{\omega}_0 P)$. Exchanging the order of differentiation and integration yields

$$\mathbf{v}(\mathbf{r}) = -\nabla \times \oint_{S'} \gamma_0 P \, dR_0 + \mathbf{v}_\infty \tag{20}$$

From equation (20), one obtains

$$\boldsymbol{\psi}(\mathbf{r}) = -\oint_{S'} \gamma_0 P \, dR_0 + \boldsymbol{\psi}_\infty(\mathbf{r}) \tag{21}$$

where $\boldsymbol{\psi}$ is the vector potential of the velocity vector and is defined by

$$\mathbf{v} = \nabla \times \boldsymbol{\psi} \tag{22}$$

and $\boldsymbol{\psi}_\infty$ is the vector potential of the free-stream velocity, $\boldsymbol{\gamma}$ is the strength of the vortex sheet, and S' is the position of the vortex sheet. With the vortex sheet representing the vorticity in the boundary layers, S^+ is a surface enclosing S and is not identical to S. As an approximation, one may consider S^+ to be external to S (and therefore in the fluid domain) and is separated from S by an infinitesimal distance.

Equation (21) is valid for both three-dimensional and two-dimensional flows. Applying equation (21) at the surface S, where the vector potential is known, one obtains an integral equation containing $\boldsymbol{\gamma}$ as the unknown function. To demonstrate the procedure for computing $\boldsymbol{\gamma}$, consider a two-dimensional flow past the exterior of a thin aerofoil at a low angle of attack. For this two-dimensional flow, equation (21) becomes

$$\psi(\mathbf{r}) = -\oint_{S'} \gamma_0 P \, dR_0 + \psi_\infty(\mathbf{r}) \tag{23}$$

where ψ is the stream function of the flow.

On the surface S, ψ vanishes and one obtains from equation (23) the following integral equation

$$\oint_{S^+} \gamma_0 P_s \, dR_0 = \psi_\infty(\mathbf{r}_s) \tag{24}$$

where $P_s = P(\mathbf{r}_s, \mathbf{r}_{s^+})$, $\mathbf{r}_s$ is a point on the solid surface S and $\mathbf{r}_{s^+}$ is a point on S^+.

The solution of equation (24), subject to an auxiliary condition, determines the vortex strength distribution on the surface S^+. In the present problem, the needed auxiliary condition is the well-known Kutta condition. Once the γ distribution is computed, it can be placed into equation (23) to compute the stream function values explicitly, point by point, everywhere in the fluid domain. This unique attribute permits the confinement of the computation field to any selected region of interest.

Consider the aerodynamic computation of the pressure distribution on the surface of the aerofoil. In order to determine the pressure, it is only necessary to know the velocity values at the outer edge of the boundary layers surrounding the aerofoil. The pressure distribution is then immediately obtained from the well-known Bernoulli's equation.

Within the boundary layer, the definition of the vorticity is simplified and becomes[4]

$$\omega = -\frac{\partial v_s}{\partial n} \tag{25}$$

where v_s is the tangential velocity components and n is the normal distance from the solid surfaces. Equation (25) gives, upon integration from the solid surface to the edge of the boundary layer,

$$\gamma = -V_{se} \tag{26}$$

where V_{se} is the velocity at the outer edge of the boundary layer. In obtaining equation (26), the no-slip condition on the solid surface is used. This equation shows that once the vortex sheet strength γ is obtained, then the boundary-layer edge velocity, and hence also the pressure on the aerofoil surface, is immediately determined. With the integral-representation procedure just described, therefore, the computation of the pressure requires only the solution of equation (20). The entire computation can be confined to the surface S^+. In consequence, for the two-dimensional potential flow problem under consideration, it is only necessary to deal with a one-dimensional integral equation.

Equation (23) can be obtained alternatively by noting that the stream function satisfy the following Poisson's equation

$$\nabla\psi^2 = -\omega \tag{27}$$

The use of equation (2) then leads to equation (23).

It is easy to see that, with the integral-representation approach, the computation of the surface pressure in a three-dimensional potential flow problem requires the solution of a two-dimensional integral equation. The region of integration of the integral equation is the solid boundary. In the special case of a flow past an axisymmetric body, the two-dimensional integral equation can be further reduced to a one-dimensional integral equation over the profile of the boundary surface in a meridian plane.[5] The advantages offered by the integral-representation approach in such computations are obviously drastic.

It should be emphasized that integral equations equivalent to equation (23) and its three-dimensional extensions have been used extensively in the computation of potential flows.[5] The derivations of the integral equations are usually based on the concepts of an inviscid fluid, and fictitious source–sink singularities are usually employed.[5] In the present

work, in contrast, the potential flow is considered an approximation of a high Reynolds number viscous flow containing no appreciable separated region. Since no fictitious singularities are employed, the use of the integral representation promotes the understanding of the interplay between the physical and the numerical aspects of the flow problem. This understanding has been an important contributor to the present development of numerical procedures for viscous flows.

BOUNDARY LAYERS

In the case of a flow containing no appreciable separated region, the vortex sheet strength, γ, determined from the integral equation, (24), represents the total vorticity across the boundary layer surrounding the solid surface. According to equation (26), the velocity at the edge of the boundary layer is equal in magnitude to the vortex strength. Thus, the computation of the vortex strength is equivalent to the computation of the boundary layer edge velocity. There exist a number of efficient and accurate computer codes with which boundary layer flows can be solved routinely and economically.

In the case of a Navier–Stokes flow, separation is an important feature and the procedure described for the potential flow computation must be extended. Wherever the flow is attached, a vortex sheet strength, or equivalently a boundary-layer edge velocity, can be computed by solving an integral equation, after which the computation of the flow within the boundary layer is routine. A detailed discussion of the generalized procedure is presented by Wu and Gulcat.[4]

NAVIER–STOKES FLOWS

A Navier–Stokes flow differs from a potential flow in that the region of non-zero vorticity in the former is not confined to thin boundary layers adjacent to solid surfaces. The vorticity distribution in the fluid cannot be represented accurately by vortex sheets alone. Considerable experience has been accumulated during the past few years by this author and his co-workers in the use of the integral representation approach for computing Navier–Stokes flows. Highly complicated Navier–Stokes flows have been solved using this approach. The discussions of this approach given in this paper are brief. Suitable references containing detailed analyses and extensive numerical results are given along with the brief discussions.

In the computation of a Navier–Stokes flow, it is convenient to follow the kinetic development of the vorticity field in the fluid. A numerical procedure can be established in which the solution is advanced from an old level to a new level through a computation loop consisting of a kinetic part and a kinematic part. This loop advances the solution by a time step in a time-dependent problem and by an iteration in a steady problem.

The kinetic part of the computation loop is concerned with the transport of vorticity in the fluid domain. In this part, with vorticity and velocity distributions known at the old level, equation (16) is solved to obtain new vorticity values in the interior of the fluid domain for the new computation level.

For steady flows, the kinetics of the problem is described by equations (14) and (15). These equations are usually solved by the use of a finite-difference procedure, or a finite-element procedure not based on the concept of fundamental solution. Alternatively, an integral representation for the vorticity vector is obtained from equation (39):

$$\boldsymbol{\omega}(\mathbf{r}) = -\frac{1}{2\pi\nu}\int_R (\mathbf{v}_0\times\boldsymbol{\omega}_0 - \nabla_0 h_0)\times\nabla_0 P\,\mathrm{d}R_0 + \oint_B [\boldsymbol{\omega}_0\cdot\mathbf{n}_0 - (\boldsymbol{\omega}_0\times\mathbf{n}_0)\times]\nabla_0 P\,\mathrm{d}B_0 \qquad (28)$$

The integral over R of ∇h can be re-expressed as a boundary integral.[11]

Procedures for computation of vorticity values in the interior of the fluid domain using the integral representation for the vorticity vector have been developed and presented.[6-8] The use of finite-element methodology yields algebraic equations approximating equation (28). There exist a large number of options in the selection of specific types and interpolation functions. The use of polygonal elements and polynomial interpolation functions are discussed in detail in Reference 7.

For time-dependent flows, the kinetics of the problem is described by equation (16). This equation is also usually solved by the use of a finite-difference or a familiar finite-element procedure. Alternatively, an integral representation for the vorticity vector is obtained by using equation (43):

$$\boldsymbol{\omega}(\mathbf{r}, t) = \int_R (\boldsymbol{\omega}_0 Q)_{t-0}\, dR_0 + \int_0^t dt_0 \int_R (\mathbf{v}_0 \times \boldsymbol{\omega}_0) \times \nabla_0 Q\, dR_0$$
$$- \nu \int_0^t dt_0 \int_B [\boldsymbol{\omega}_0 (\nabla_0 Q \cdot \mathbf{n}_0) - Q_0 (\mathbf{n}_0 \cdot \nabla_0) \boldsymbol{\omega}_0]\, dB_0 \tag{29}$$

The principal solution Q gives the vorticity distribution in an infinite unlimited stationary fluid at the time level t as a result of diffusion of a unit amount of vorticity located at $\mathbf{r}_0$ at a preceding time level t_0. The first integral in equation (29) therefore represents the effect of an initial ($t = 0$) vorticity distribution. In actual flows, a stationary fluid cannot co-exist with a non-zero vorticity field. The vorticity distribution changes as a result of not only the diffusion process but also the convection and the vortex stretching processes. The cumulative effects of these additional processes on the vorticity distribution at the time level t are represented by the second integral. The third integral gives the effect of the boundary values of $\boldsymbol{\omega}$ on the vorticity distribution in R at the time level t. This integral may be viewed as representing the effect of diffusion of the vorticity field outside the region R on the vorticity field in the region R.

The use of the finite-element methodology in the numerical quadratures with respect to time and space yields algebraic equations approximating equation (29). There exist again a large number of options in the selection of specific element types and interpolation functions, with respect to both time and space. The algebraic equations used by the present author and his co-workers are described in References 9–11.

The distinguishing feature of the integral representation, equation (29), is that it expresses the several kinetic processes that redistribute the vorticity in the fluid as separate integrals. This feature offers the possibility of using different numerical quadrature procedures for the evaluation of the different kinetic processes. In this manner, the drastically different characteristic time scales of these kinetic processes can be individually accommodated. Also, as noted before, the contribution of the initial vorticity distribution in R to the distribution at a later time is identical to the contribution of the diffusion process in an infinite unbounded region. In evaluatiing the first integral in equation (29), therefore, one needs to solve only the homogeneous diffusion equation in an infinite unbounded region. The form of the principal solution Q, which appears in each of the three integrals of equation (29), indicates that interpolation functions other than the familiar polynomials should be used for both the time element and the space elements. It appears that simple efficient and accurate procedures can be developed on the basis of these attributes.

In order to carry out the kinematic part of the computation and to advance the solution further, boundary values of vorticity must be determined. These values are not ordinarily prescribed directly from the physics of the problem. Moreover, these values are not determined by the kinetics of the problem. Rather, they are governed by the kinematics of

the problem, as shall be explained shortly. The kinematic part of the computation loop is therefore composed of two steps: (a) the computation of the boundary vorticity values and (b) the computation of velocity values in the interior of the fluid domain. The velocity values at the boundary of the fluid domain are considered to be prescribed and need not be computed.

It has been shown that,[12] with any given distribution of vorticity in a singly connected region R and on its boundary B, if either the normal component or the tangential component of the velocity is specified on B, then a unique solution of equations (9) and (11) exists in R. If R is multiply connected, then the circulations around the several independent circuits in R must also be specified. The specification of both the normal and the tangential components of velocity on B overspecifies the problem. It should be pointed out that both the normal and the tangential velocity components are prescribed by the physics of the problem; and both appear in the boundary integral of equation (17). The use of equation (17) therefore appears to overspecify the kinematics of the problem. In reality, the prescription of both the normal and the tangential velocity component places a kinematic restriction on the distribution of vorticity in R. This restriction permits the boundary vorticity values to be determined.

In Reference 12, detailed discussions of the use of equation (17) in the computation of the boundary vorticity values are presented. The procedure described in Reference 12 can be considered an extension of the procedure described earlier for the computation of potential flows. For the Navier–Stokes flow being considered, the vorticity in a thin layer surrounding the solid surface can be represented by a vortex sheet of strength γ. For two-dimensional flows, a derivation similar to that preceding equation (23) yields an integral representation:

$$\psi(\mathbf{r}) = -\int_{S} \gamma_0 P \, dR_0 - \int_{R^-} \omega_0 P \, dR_0 + \psi_\infty(\mathbf{r}) \tag{30}$$

where R^- is the region outside of the thin layer surrounding the solid surface.

Equation (30) differs from equation (23) only in the presence of the integral over R^-. Since R^- is the interior of the fluid domain, the vorticity in R^- is determined kinetically from the vorticity transport equation. Equation (30), when applied to the solid surface S, gives

$$\int_{S} \gamma_0 P_s \, dR_0 = \psi(\mathbf{r}_s) + \psi_a(\mathbf{r}_s) \tag{31}$$

where ψ_a is the contribution of the vorticity in R^- to the stream function on S. Since ψ_a is known, in the kinematic part of the computation the vortex sheet strength γ can be determined by solving equation (31). In the attached part of the Navier–Stokes flow, γ gives the boundary-layer edge velocity in accordance with equation (26). In the separated part of the Navier–Stokes flow, the vortex sheet strength γ represents the total vorticity across a thin layer in the separated flow. The boundary value of vorticity can be determined by distributing the total vorticity suitably across the layer.

The boundary vorticity values can be computed kinematically without relying upon the concept of a vortex sheet. To accomplish this, one uses the finite-element methodology and approximates equation (17) by

$$\mathbf{v}_m = \sum_{n=1}^{N} \mathbf{A}_{mn} \times \boldsymbol{\omega}_n + \sum_{b=1}^{B} (B_{mb}\mathbf{v}_b + \mathbf{C}_{mb} \times \mathbf{v}_b) \tag{32}$$

where the subscripts m, n and b designate, respectively, velocity nodes, vorticity nodes and velocity nodes on B; A_{mn}, B_{mb} and C_{mb} are geometrical coefficients depending only upon the

relative positions of the nodes m and n, or m and b; N is the total number of nodes in R and on B; B is the number of nodes on B.

Applying equation (32) at nodes on B yields

$$\mathbf{v}_c = \sum_{n=1}^{N} \mathbf{A}_{cn} \times \boldsymbol{\omega}_n + \sum_{b=1}^{B} (B_{cb}\mathbf{v}_b + \mathbf{C}_{mb} \times \mathbf{v}_b) \tag{33}$$

where the subscript c indicates a boundary velocity node.

In equation (33), the only unknown quantities are the boundary values of $\boldsymbol{\omega}$ on B. A solution of equation (33) yields the boundary values of $\boldsymbol{\omega}$.

Once the boundary values of $\boldsymbol{\omega}$ are computed, equation (32) can be used to compute $\mathbf{v}$ values explicitly, point by point. The unique attribute of the integral representation approach in explicitly computing the velocity values offers great advantages to the computation of Navier–Stokes flows. It permits the velocity values to be computed in any selected region of the flowfield. It is obvious from the vorticity transport equation (16) that information about the velocity field is needed only in the region of non-zero vorticity in order to determine the effect of convection. This region of non-zero vorticity is the only region in which viscous effects are important. In many viscous flow problems, including those involving massive regions of separation, the viscous region occupies only a small portion of the total flow field. The integral representation permits the confinement of the computation to the viscous region. In consequence, drastic reductions in the number of data points and in the amount of computation was achieved.

The integral representation further permits the confined solution field to be segmented and the computation within each segment performed independently of those in other segments.[13] It also permits the boundary layer region of the flow to be treated separately from the detached viscous region.[4,14] Furthermore, it is simple to form hybrid approaches[15,16] in which the integral representation approach is used advantageously in some parts of the flow.

The ability to treat the boundary layer and the detached regions of the Navier–Stokes flows separately is of particular importance to the computation of high Reynolds number flows. In such flows, because of the drastic difference in the length scales associated with the boundary layers and the detached regions, the computer time requirements is often excessive and the solution accuracy is usually inadequate.[4,17] This drastic difference in length scale is, in fact, responsible for the Reynolds number limit of previous numerical methods. This limit is successfully removed by a separate treatment of the boundary layer and detached regions.[4,14,17]

ILLUSTRATIVE RESULTS

It has been found that the integral-representation approach is well-suited for a wide range of viscous flow problems involving flow separation. In recent articles, this author and his co-workers have presented numerical results for turbulent flows,[18,19,20] for compressible flows,[21,22] and for internal steady flows.[6,7,8] In these recent works, integral representations are used in the kinematic part of the computation. Rizk,[10] however, obtained results for viscous flows using integral representations in both the kinetic and kinematic parts of the computation. Selected results for Navier–Stokes flows are presented here to illustrate the application of the integral-representation approach.

The flow past a circular cylinder at a Reynolds number based on the cylinder diameter of 40,000 is computed using the integral-representation approach and treating the boundary layer and the detached regions in a Navier–Stokes flow separately. In Figure 1 are shown the pressure distributions computed using the integral-representation approach compared with

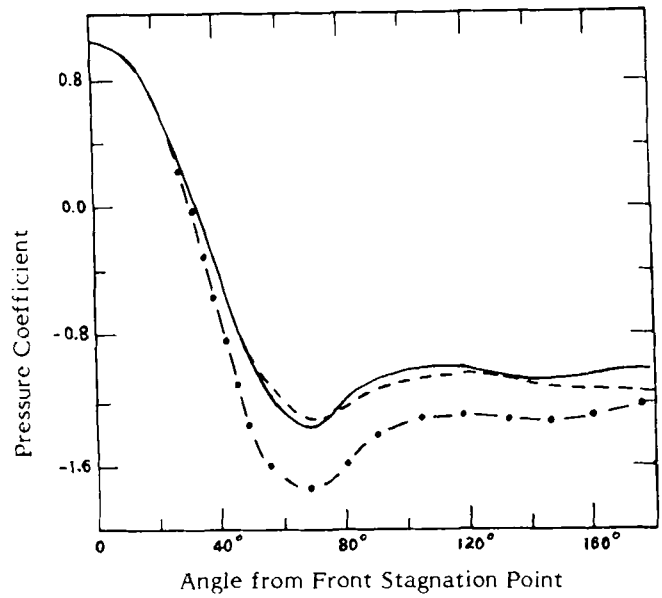

Figure 1. Surface pressure distribution on a circular cylinder at a Reynolds number of 40,000: —— experimental data; - - - integral representation; - . - finite difference

experimental data[23] and finite-difference results.[24] The agreement between the present results and the experimental data is remarkable. In contrast, the finite-difference results deviate substantially from the experimental data. The accuracy of the integral-representation approach has been also calibrated by computing flows past circular cylinders at lower Reynolds numbers. In all cases computed, the agreements between the present results and experimental data are excellent.[9]

A most reassuring feature of the integral-representation approach is that the computer time required to solve a given flow problem is insensitive to the flow Reynolds number. At the present, the computation of the flow past the circular cylinder requires approximately twenty minutes of CDC-6600 CPU time for laminar flows in the Reynolds number range of 500 to 100,000. The integral-representation approach is not Reynolds number limited.

In Figure 2 is shown a computed turbulent flow pattern past a 12 percent symmetric

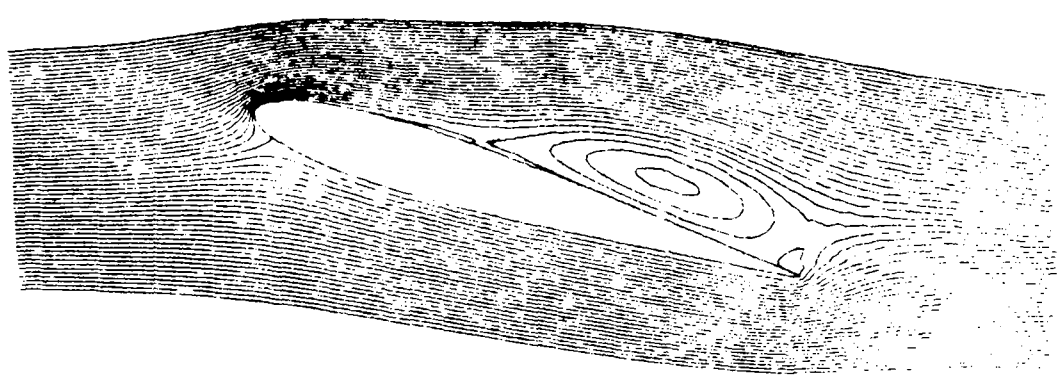

Figure 2. Streamline around 12 percent thick aerofoil at 15° angle of attack and Reynolds number of $3{\cdot}6 \times 10^6$

aerofoil at an angle of attack of 15° and at a Reynolds number based on the chord length of $3{\cdot}6\times10^6$. A two-equation differential model of turbulence is used.[19,20] The aerofoil is set into motion from rest and the solution is carried to a large time level. The flow field around the aerofoil does not approach an asymptotic steady state. Rather, a cyclic shedding of vortices from the separated region near the aerofoil occurs. This cyclic shedding of vortices is similar to the well-known Karman vortex shedding behind a circular cylinder.

CONCLUDING REMARKS

The integral-representation procedure that has been developed on the basis of principal solutions represents a major departure from previous finite-difference and finite-element methods. The use of the integral representation containing the principal solutions offer a number of decisive advantages. The most important advantage is the removal of the Reynolds number limit in the computation of Navier–Stokes flows. Highly efficient computer codes have been prepared for the computation of two-dimensional laminar and turbulent, steady and time-dependent, incompressible and compressible viscous flows involving large separation regions in two dimensions. Current efforts are directed towards the full use of the advantageous attributes of the integral representation for the kinetic aspect of viscous flows and the development of highly efficient three-dimensional algorithms.

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APPENDIX

An integral representation for the scalar Poisson's equation is given in the main text of this paper as equation (2). Integral representations for two other types of differential equations relevant to flow problems are given here.

(a) *Vector field with specified dilatation and rotation.*

Consider the pair of vector equations:

$$\nabla.\mathbf{F} = g, \qquad \nabla\times\mathbf{F} = \mathbf{G} \tag{34}$$

where g and $\mathbf{G}$ are, respectively, the dilatation and rotation of $\mathbf{F}$ and are known functions of position and field variables (including $\mathbf{F}$) and are continuous and bounded in the region R. The vector $\mathbf{F}$ may be decomposed into a solenoidal part $\mathbf{F}_1$ and an irrotational part $\mathbf{F}_2$,

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 \tag{35}$$

with $\mathbf{F}_1$ satisfying

$$\nabla.\mathbf{F}_1 = 0, \qquad \nabla\times\mathbf{F}_1 = \mathbf{G}, \tag{36}$$

and $\mathbf{F}_2$ satisfying

$$\nabla.\mathbf{F}_2 = g, \qquad \nabla\times\mathbf{F}_2 = 0 \tag{37}$$

Since $\mathbf{F}_2$ is irrotational, a scalar potential ϕ, defined by $\nabla\phi = \mathbf{F}_2$, exists. The first equation of (37) therefore gives $\nabla^2\phi = g$. An integral representation for $\mathbf{F}_2$ is given by equation (2). By taking the gradient of equation (2), one obtains an integral representation for $\mathbf{F}_2$. Taking the curl of the second equation in (36) and using the first equation in (36), one obtains a vector

Poisson's equation $\nabla^2\mathbf{F}_1 = -\nabla\times\mathbf{G}$. An integral representation for $\mathbf{F}_1$ is then obtained by replacing the scalar quantities ϕ and g in equation (2) by the vector quantities $\mathbf{F}$ and $-\nabla\times\mathbf{G}$. For computational purposes, however, it is more convenient to consider the pair of equations $\nabla\times\boldsymbol{\psi} = \mathbf{F}_1$, where $\boldsymbol{\psi}$ is the vector potential whose existence is implied by the fact $\mathbf{F}_1$ is solenoidal, and $\nabla\times\mathbf{F}_1 = \mathbf{G}$. This pair of equations is equivalent to (36). The vector fundamental solution $\mathbf{P}$ for the pair is

$$\mathbf{P} = \nabla P\times\mathbf{a} \tag{38}$$

where P is the scalar fundamental solution defined by equation (3). By the use of P, an integral representation for $\mathbf{F}_1$ is obtained.[9] Adding the integral representation for $\mathbf{F}_1$ to that for $\mathbf{F}_2$, one obtains

$$\mathbf{F}(\mathbf{r}) = -\int_R (g_0 + \mathbf{G}_0\times)\nabla_0 P\,\mathrm{d}R_0 + \int_B [\mathbf{F}_0\cdot\mathbf{n}_0 - (\mathbf{F}_0\times\mathbf{n}_0)\times]\nabla_0 P\,\mathrm{d}B_0 \tag{39}$$

which is an integral representation for $\mathbf{F}$ corresponding to the pair of vector equations (34)

(b) *Transport equations*

The time-dependent transport equation is parabolic in its time-space relationship and expressible in the scalar case in the form

$$\nabla^2\psi - \frac{1}{a}\frac{\partial\psi}{\partial t} = \phi \tag{40}$$

where ϕ may be a function of position, time, ψ and first spatial derivatives of ψ and of other field variables. For the case when a is a constant, the fundamental solution corresponding to equation (40) is

$$Q(\mathbf{r}, t; \mathbf{r}_0, t_0) = \frac{1}{(4\pi at)d/2}\exp\left(\frac{r'^2}{4at'}\right) \tag{41}$$

where $t' = t_0 - t$ and d is the number of spatial dimensionality; i.e. $d = 3$ for problems involving three spatial dimensions, etc. The principal solution possesses the following properties:

$$\nabla^2 Q + \frac{1}{a}\frac{\partial Q}{\partial t} = 0, \quad \text{in } R \text{ for all } t \tag{42}$$

and for $t = t_0$

$$\int_R Q\,\mathrm{d}R = \begin{cases} 0, & \text{for } R \text{ not containing } \mathbf{r} = \mathbf{r}_0 \\ 1, & \text{for } R \text{ containing } \mathbf{r} = \mathbf{r}_0 \end{cases}$$

By integrating the expression

$$Q\left(\nabla^2\psi - \frac{1}{a}\frac{\partial\psi}{\partial t}\right) - \psi\left(\nabla^2 Q + \frac{1}{a}\frac{\partial Q}{\partial t}\right)$$

over R and over the time interval $0 < t < t_0$ and using the divergence theorem, one obtains

$$\begin{aligned}\psi(\mathbf{r}, t) = &-a\int_0^t \mathrm{d}t_0\int_R Q\phi_0\,\mathrm{d}R_0 + \int_R (Q\psi_0)_{t_0=0}\,\mathrm{d}R_0 \\ &+ a\int_0^t \mathrm{d}t_0\int_B (Q\nabla_0\psi_0 - \psi_0\nabla_0 Q)\cdot\mathbf{n}_0\,\mathrm{d}R_0\end{aligned} \tag{43}$$

Equation (43) is an integral representation for ψ corresponding to the scalar transport equation (40). For a vector transport equation, an integral representation is readily obtainable from equation (43) by replacing ψ and ϕ by their vector counterparts in the vector transport equation.

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